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LETTER TO THE EDITOR

ON THE REAL EIGENVALUES FOR VIBRATING STRUCTURES COUPLED WITH QUIESCENT LIQUIDS WITH FREE SURFACE

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Recently, Amabili (1997, 1999), Amabili *et al.* (1998), Gonçalves & Ramos (1996), Chiba (1993, 1995) and Chiba & Osumi (1998) solved vibration of plates and shells coupled with sloshing, quiescent and inviscid liquid by inserting the sloshing condition into the eigenvalue problem. In their formulation, all the authors cited obtained an eigenvalue problem for non-symmetric matrices for which the problem of the existence of complex eigenvalues arises. As a matter of fact a general criterion for the reality of the eigenvalues does not exist, while for symmetric matrices this property is always verified.

However, different variational approaches developed for Finite Element codes obtain eigenvalue problems for symmetric matrices (e.g., Balendra *et al.* 1981; Zienkiewicz & Taylor 1991; Morand & Ohayon 1995) that give real eigenvalues. The aim of the present letter is to clarify this apparent contradiction.

Using the analysis and the symbols introduced by Amabili (1997), for the vibration of a thin-walled structure coupled to a sloshing, incompressible and inviscid fluid, the Galerkin equation reads as follows:

$$\begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{K}_1 & \mathbf{K}_2 \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{h} \end{Bmatrix} - \omega_r^2 \begin{bmatrix} \mathbf{M} + \mathbf{M}_a & \mathbf{M}_S \\ \mathbf{0} & \mathbf{M}_1 \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{h} \end{Bmatrix} = 0, \tag{1}$$

where $\mathbf{q}$ and $\mathbf{h}$ are the generalized coordinates obtained by the discretization of the system. In equation (1), the vector $\mathbf{q}$ is associated with bulging modes and the vector $\mathbf{h}$ with sloshing modes. Bulging and sloshing modes are coupled by the matrices $\mathbf{M}_S$ and $\mathbf{K}_1$. In particular, $\mathbf{M}_S$ is the added mass matrix associated with the reference kinetic energy due to the sloshing of the fluid and it is given by

$$\rho_F \iint_{S_0} \Phi_S \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S = \mathbf{q}^{\mathrm{T}} \mathbf{M}_S \mathbf{h}. \tag{2}$$

The matrices $\mathbf{K}_1$, $\mathbf{K}_2$ and $\mathbf{M}_1$ come from the vectorial form of the sloshing equation that is inserted in the eigenvalue problem. In particular, all the terms of the sloshing equation can

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be multiplied by $\rho_F \Phi_S \, \mathrm{d}S$ and integrated over the free surface S_F in order to give an algebraic equation. This operation gives

$$\rho_F \iint_{S_F} \Phi_S(\partial \Phi_B/\partial n)\, \mathrm{d}S = \mathbf{h}^{\mathrm{T}}\mathbf{K}_1\mathbf{q}, \quad \rho_F \iint_{S_F} \Phi_S(\partial \Phi_S/\partial n)\, \mathrm{d}S = \mathbf{h}^{\mathrm{T}}\mathbf{K}_2\mathbf{h}, \tag{3a, b}$$

$$(\rho_F/\mathrm{g}) \iint_{S_F} (\Phi_S)^2 \, \mathrm{d}S = \mathbf{h}^{\mathrm{T}}\mathbf{M}_1\mathbf{h}. \tag{3c}$$

Equation (1) gives an eigenvalue problem for a real, nonsymmetric matrix. The same Galerkin equation was obtained by Gonçalves & Ramos (1996), using the Galerkin method, and by Amabili (1999) and Amabili *et al.* (1998) by using the Rayleigh–Ritz method. Chiba (1993, 1995) and Chiba and Osumi (1998) obtained another nonsymmetric Galerkin equation. It can easily be shown that equation (1) can give complex eigenvalues. Let us consider for simplicity the case when only one sloshing and one bulging mode are retained in the Rayleigh–Ritz expansion. In this case the eigenvalue problem has dimension 2×2 and has complex conjugate eigenvalues when

$$\left(\frac{\omega_B - \omega_S}{\omega_B}\right)^2 < \frac{M_S K_1}{M_1 K} < \left(\frac{\omega_B + \omega_S}{\omega_B}\right)^2. \tag{4}$$

In equation (4) $\omega_B^2 = K/(M + M_a)$ is the squared radian frequency of the bulging mode neglecting free surface waves and $\omega_S^2 = K_2/M_1$ is the squared radian frequency of the sloshing mode in the rigid tank.

The question of complex eigenvalues is actually a false problem. In fact, it can be proved that

$$\mathbf{K}_1 = -\mathbf{M}_S^{\mathrm{T}}. \tag{5}$$

In the particular case of dimension 2, we can see that condition (5) is in contradiction with relation (4) and the existence of complex eigenvalues is completely excluded. Equation (5) allows simplified computations of the matrix coupling the sloshing and bulging modes. Both the expressions for $\mathbf{M}_S$ and $\mathbf{K}_1$ can be used; however, the simplest one, which depends on the problem under investigation, will be used here. Equation (5) is equivalent to

$$\rho_F \iint_{S_F} \Phi_S(\partial \Phi_B/\partial n)\, \mathrm{d}S = -\rho_F \iint_{S_0} \Phi_S \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S. \tag{6}$$

The property expressed by equation (6) is a direct consequence of the following relationship between two distinct modes of the irrotational fluid:

$$\iint_{\partial V} \phi \frac{\partial \phi'}{\partial n} \mathrm{d}S = \iint_{\partial V} \phi' \frac{\partial \phi}{\partial n} \mathrm{d}S, \tag{7}$$

where ∂V indicates the boundary of the fluid volume and Φ_S is taken for ϕ and Φ_B for ϕ', which is a straightforward application of Green's theorem. Equation (6) is immediately obtained from equation (7) as a consequence of the fact that $\iint_{\partial V} \Phi_B(\partial \Phi_S/\partial n)\, \mathrm{d}S = 0$, for the boundary conditions assumed by Amabili (1997).

By using equation (5), equation (1) can be transformed into a Galerkin equation for symmetric matrices with simple manipulations (Balendra *et al.* 1982). The final Galerkin equation for symmetric matrices is

$$\begin{bmatrix} \mathbf{K} + \mathbf{M}_S\mathbf{M}_1^{-1}\mathbf{M}_S^{\mathrm{T}} & -\mathbf{M}_S\mathbf{M}_1^{-1}\mathbf{K}_2 \\ -\mathbf{K}_2\mathbf{M}_1^{-1}\mathbf{M}_S^{\mathrm{T}} & \mathbf{K}_2\mathbf{M}_1^{-1}\mathbf{K}_2 \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{h} \end{Bmatrix} - \omega_r^2 \begin{bmatrix} \mathbf{M} + \mathbf{M}_a & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_2 \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{h} \end{Bmatrix} = 0. \tag{8}$$

In conclusion, real eigenvalues ω_r^2 are associated with equation (1). The advantages of equation (5) are evident. Details of the present analysis will be given in a subsequent paper (Amabili 2000).

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